On 4-Dimensional Hypercomplex Algebras in Adaptive Signal Processing

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Abstract. The degree of diffusion of hypercomplex algebras in adaptive and non-adaptive filtering research topics is growing faster and faster. The debate today concerns the usefulness and the benefits of representing multidimensional systems by means of these complicated mathematical structures and the criterions of choice between one algebra or another. This paper proposes a simple comparison between two isodimensional algebras (quaternions and tessarines) and shows by simulations how different choices may determine the system performance. Some general information about both algebras is also supplied.

Keywords: Adaptive Filters, Quaternions, Tessarines, Hypercomplex, Widely Linear, Least Mean Square

1 Introduction

One of the trends in the last two decades in digital signal processing has been the exploration of hypercomplex algebras for multidimensional signal processing with particular regard to adaptive filtering and intelligent systems. Since complex numbers were widely experimented and studied in both linear and nonlinear environments [9], the immediate step forward in hypercomplex algebras has considered quaternions [19] and octonions [6]. Scientists paid special attention to quaternion adaptive filtering [4, 5, 7, 17, 19] and the authors of this paper themselves investigated quaternion algebra with an interest in frequency-domain adaptive filters [11]. Besides the traditional usage of quaternions in 3D graphics and navigation (it is known that quaternion rotations are not subject to deadlocks from space degeneration from 3D to 2D) [18], the new revival of hypercomplex processing consists in the experimentation of the peculiar algebraic properties in engineering multidimensional problems. Multidimensionality is somehow intrinsic to the nature of the data: it arises from the need for processing correlated data (not necessarily homogeneous, as in [19]).

The investigation topic in this paper considers two hypercomplex algebras having the same dimensions: quaternions and tessarines (the latter also known as *bicomplex numbers*). Both of them are a 4-dimensional algebra. So, which are the reasons why we should choose one algebra or another? This paper presents a couple of examples where different results are obtained from different mathematical representations of the systems. A line of reasoning is also suggested. With regard to adaptive signal processing, the first adaptive algorithm implemented in a hypercomplex algebra has been the Quaternion Least Mean Square (QLMS) algorithm [19]. Because of its straight comprehensibility and ease of implementation, this algorithm offered an instrument on hand for studying quaternion algebra combined with adaptive filtering. In this work, we adopted this algorithm to make a comparison of the two 4-dimensional algebras above mentioned. On this occasion, we derived and implemented a tessarine version of the LMS algorithm, namely TLMS. In a second step, we searched for a modification of both the QLMS and TLMS algorithms into a *widely linear* form (including full second order statistics) [8, 20, 21] and their behaviour was tested with *proper* and *improper* input signals. The highlight in this paper is the evidence that the choice of a specific algebra may condition a filter behaviour. We analysed this fact by introducing Ambisonic 3D audio signals into a 4-dimensional system. Ambisonics is a 3D audio recording and rendering technique developed in the '70s [2, 3, 16] and, in recent research, it was experimented that its so-called B-Format can be condensed and processed in a quaternion formalism [12, 13]. The main goal of the current work is to examine whether the good results obtained with quaternion algebra persist in other 4-dimensional algebras.

This paper is organized as follows: Section 2 introduces both quaternion and tessarine algebras underlining the fundamental differences between them; Section 3 presents a short summary of the 4-dimensional least mean square algorithms (QLMS and TLMS) and their *widely linear* modifications. Finally, Section 4 reports some interesting results from simulations with both widely linear and non-widely linear algorithms.

2 Introduction to 4-dimensional hypercomplex algebras

As just introduced, both quaternions and tessarines are 4-dimensional algebras. Even though a quaternion q and a tessarine t look just alike ($q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ with $q \in \mathbb{H}$ and $t = t_0 + t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$ with $t \in \mathbb{T}$), their algebras have very little in common. The imaginary axes ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) form a basis on which the two algebras are built. However, in the two cases, different fundamental algebraic properties are defined:

In quaternion algebra,

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}, \tag{1}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1. \tag{2}$$

In tessarine algebra,

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{j}, \tag{3}$$

$$\mathbf{i}^2 = \mathbf{k}^2 = -1, \quad \mathbf{j}^2 = +1. \tag{4}$$

Equations (1)–(4) embody the foremost difference between quaternion and tessarine algebras: the former is non-commutative, the latter is commutative. As a consequence of this, quaternion product and tessarine product are defined in different ways.

For quaternions q_1, $q_2 \in \mathbb{H}$, their product is calculated as

$$\begin{aligned} q_1 q_2 &= (a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\,(b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) \\ &+ (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)\,\mathbf{i} \\ &+ (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1)\,\mathbf{j} \\ &+ (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\,\mathbf{k}. \end{aligned} \tag{5}$$

For tessarines t_1, $t_2 \in \mathbb{T}$, their product is calculated as

$$\begin{aligned} t_1 t_2 &= (a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\,(b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= (a_0b_0 - a_1b_1 + a_2b_2 - a_3b_3) \\ &+ (a_0b_1 + a_1b_0 + a_2b_3 + a_3b_2)\,\mathbf{i} \\ &+ (a_0b_2 - a_1b_3 + a_2b_0 - a_3b_1)\,\mathbf{j} \\ &+ (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)\,\mathbf{k}. \end{aligned} \tag{6}$$

Moreover, (1) can be expressed by the *cross products* $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$. In fact, the cross product is non-commutative (anti-commutative).
Regarding the sum, it is computed the same way with either quaternions or tessarines:

$$\begin{aligned} q \pm p &= (q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) \pm (p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}) \\ &= (q_0 \pm p_0) + (q_1 \pm p_1)\,\mathbf{i} + (q_2 \pm p_2)\,\mathbf{j} + (q_3 \pm p_3)\,\mathbf{k}. \end{aligned} \tag{7}$$

The *poly-conjugation*, with respect to the specific algebraic rules, conveniently defines both the conjugates of a quaternion and a tessarine:

$$p^* = p_0 + \sum_{\nu=1}^{3} p_\nu \mathbf{e}_\nu^3 \tag{8}$$

and typified in the two algebras becomes $q^* = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k} \in \mathbb{H}$ and $t^* = t_0 - t_1\mathbf{i} + t_2\mathbf{j} - t_3\mathbf{k} \in \mathbb{T}$.

3 4D Least Mean Square algorithms

In the 4-dimensional algebra of quaternions, the least mean square algorithm was formerly presented in [19] with a minor modification in [1]. Following an approach similar to [19], the TLMS algorithm has been derived by the authors of this paper. This section compares and comments QLMS and TLMS.

3.1 Algorithm overview

The least mean square algorithm is an online error-correction-based adaptive algorithm: the cost function to be minimized during the adaptation is defined as the mean square error (MSE) $J_n(\mathbf{w}_n) = E\{e[n]e^*[n]\}$. The error $e[n]$ is the difference between a desired signal and the filter output ($e[n] = d[n] - y[n]$). The adaptive filter output can be defined by the scalar product $y[n] = \mathbf{w}_{n-1}^T \mathbf{x}_n$, thus resulting in

$$y[n] = \mathbf{w}_{n-1}^T \mathbf{x}_n = \begin{bmatrix} \mathbf{w}_a^T \mathbf{x}_a - \mathbf{w}_b^T \mathbf{x}_b - \mathbf{w}_c^T \mathbf{x}_c - \mathbf{w}_d^T \mathbf{x}_d \\ \mathbf{w}_a^T \mathbf{x}_b + \mathbf{w}_b^T \mathbf{x}_a + \mathbf{w}_c^T \mathbf{x}_d - \mathbf{w}_d^T \mathbf{x}_c \\ \mathbf{w}_a^T \mathbf{x}_c + \mathbf{w}_c^T \mathbf{x}_a + \mathbf{w}_d^T \mathbf{x}_b - \mathbf{w}_b^T \mathbf{x}_d \\ \mathbf{w}_a^T \mathbf{x}_d + \mathbf{w}_d^T \mathbf{x}_a + \mathbf{w}_b^T \mathbf{x}_c - \mathbf{w}_c^T \mathbf{x}_b \end{bmatrix} \in \mathbb{H} \tag{9}$$

for quaternions, and

$$y[n] = \mathbf{w}_{n-1}^T \mathbf{x}_n = \begin{bmatrix} \mathbf{w}_a^T \mathbf{x}_a - \mathbf{w}_b^T \mathbf{x}_b + \mathbf{w}_c^T \mathbf{x}_c - \mathbf{w}_d^T \mathbf{x}_d \\ \mathbf{w}_a^T \mathbf{x}_b + \mathbf{w}_b^T \mathbf{x}_a + \mathbf{w}_c^T \mathbf{x}_d + \mathbf{w}_d^T \mathbf{x}_c \\ \mathbf{w}_a^T \mathbf{x}_c + \mathbf{w}_c^T \mathbf{x}_a - \mathbf{w}_d^T \mathbf{x}_b - \mathbf{w}_b^T \mathbf{x}_d \\ \mathbf{w}_a^T \mathbf{x}_d + \mathbf{w}_d^T \mathbf{x}_a + \mathbf{w}_b^T \mathbf{x}_c + \mathbf{w}_c^T \mathbf{x}_b \end{bmatrix} \in \mathbb{T} \tag{10}$$

for tessarines, where $\mathbf{x}_n$ is the filter input vector at iteration n and $\mathbf{w}_{n-1}$ are the filter weights at iteration $n-1$: $\mathbf{x}_n = \left[\, x[n]\; x[n-1] \cdots x[n-M] \,\right]^T$, $\mathbf{w}_n = \left[\, w_0[n]\; w_1[n] \cdots w_M[n] \,\right]^T$, with M the filter length.

The computation of the gradient of the cost function $J_n(\mathbf{w}_n)$ (required for finding its minimum) leads to an adaptation equation which is the same for both QLMS and TLMS for outputs defined as in (9) and (10):

$$\mathbf{w}_n = \mathbf{w}_{n-1} + \mu e[n]\, \mathbf{x}_n^*. \tag{11}$$

where μ is the step size along the direction of the gradient.

3.2 Widely linear modification

Recent works about both complex and hypercomplex filtering showed a particular interest in widely linear algorithms [8, 10, 21, 22]. It has been observed that most real world signals are *improper* (or *noncircular*) in nature [14] and, in this case, a filter performance can be improved significantly if the full second order statistics of the signals is taken into account and included into the algorithm. If a random variable has a rotation-invariant probability distribution (with respect to all six pairs of rotation axes $(\mathbf{1},\mathbf{i})$,$(\mathbf{1},\mathbf{j})$,$(\mathbf{1},\mathbf{k})$, $(\mathbf{i},\mathbf{j})$, $(\mathbf{k},\mathbf{j})$, $(\mathbf{k},\mathbf{i})$), it must be considered *proper*, or second-order circular. Signal properness in $\mathbb{H}$ can be checked by considering that for a proper quaternion random variable $q = q_a + q_b\mathbf{i} + q_c\mathbf{j} + q_d\mathbf{k}$ the following properties hold [21]:

1. $E\{q_m^2\} = \sigma^2$, $\forall m = a, b, c, d$ (all four components of q have equal power).
2. $E\{q_m q_n\} = 0$, $\forall m, n = a, b, c, d$ and $m \neq n$ (all four components of q are uncorrelated).

3. $E\{qq\} = -2E\{q_m^2\} = -2\sigma^2, \ \forall m = a, b, c, d$ (the pseudocovariance matrix does not vanish)
4. $E\{|q|^2\} = 4E\{q_m^2\} = 4\sigma^2, \ \forall m = a, b, c, d$ (the covariance of a quaternion variable is the sum of the covariances of all components).

Since for a quaternion proper signal the complementary covariance matrices, defined as $\mathcal{C}_{\mathbf{q}}^{\mathbf{i}} = E\{\mathbf{q}\mathbf{q}^{\mathbf{i}H}\}, \mathcal{C}_{\mathbf{q}}^{\mathbf{j}} = E\{\mathbf{q}\mathbf{q}^{\mathbf{j}H}\}, \mathcal{C}_{\mathbf{q}}^{\mathbf{k}} = E\{\mathbf{q}\mathbf{q}^{\mathbf{k}H}\}$, vanish, widely linear algorithms incorporate and exploit this second order information on purpose. We need to define the quaternion *involutions* here:

$$\begin{aligned} q^{\mathbf{i}} &= -\mathbf{i}q\mathbf{i} = q_a + \mathbf{i}q_b - \mathbf{j}q_c - \mathbf{k}q_d \\ q^{\mathbf{j}} &= -\mathbf{j}q\mathbf{j} = q_a - \mathbf{i}q_b + \mathbf{j}q_c - \mathbf{k}q_d \\ q^{\mathbf{k}} &= -\mathbf{k}q\mathbf{k} = q_a - \mathbf{i}q_b - \mathbf{j}q_c + \mathbf{k}q_d. \end{aligned} \tag{12}$$

Involutions are functions $f(.)$ chosen in a way that, given $q, p \in \mathbb{H}$, we have the conditions: 1) $f(f(q)) = q$, 2) $f(q+p) = f(q) + f(p)$ and $f(\lambda q) = \lambda f(q)$, 3) $f(qp) = f(q)f(p)$.

The WL-QLMS algorithm updates four sets of filter weights: $\mathbf{w}$, $\mathbf{h}$, $\mathbf{u}$, $\mathbf{v}$ $\in \mathbb{H}^{M\times 1}$, where M is the filter length. Accordingly, the filter output is computed by convoluting each weight vector with its corresponding input involution:

$$y_w[n] = \mathbf{w}_{n-1}^T \mathbf{x}_n, \quad y_h[n] = \mathbf{h}_{n-1}^T \mathbf{x}^{\mathbf{i}}{}_n, \quad y_u[n] = \mathbf{u}_{n-1}^T \mathbf{x}^{\mathbf{j}}{}_n, \quad y_v[n] = \mathbf{v}_{n-1}^T \mathbf{x}^{\mathbf{k}}{}_n \tag{13}$$

and summing all four contributions:

$$y[n] = y_w[n] + y_h[n] + y_u[n] + y_v[n]. \tag{14}$$

In conclusion, we have four adaptation equations:

$$\begin{aligned} \mathbf{w}_n &= \mathbf{w}_{n-1} + \mu e[n]\,\mathbf{x}_n^*, \quad & \mathbf{h}_n &= \mathbf{h}_{n-1} + \mu e[n]\,\mathbf{x}_n^{\mathbf{i}*} \\ \mathbf{u}_n &= \mathbf{u}_{n-1} + \mu e[n]\,\mathbf{x}_n^{\mathbf{j}*}, \quad & \mathbf{v}_n &= \mathbf{v}_{n-1} + \mu e[n]\,\mathbf{x}_n^{\mathbf{k}*}. \end{aligned} \tag{15}$$

Is it possible to obtain a similar algorithm with tessarines? Well, if we apply the three conditions above in order to find tessarine involutions we obtain the following results ($t \in \mathbb{T}$):

$$\begin{aligned} t^{\mathbf{i}} &= -\mathbf{i}t\mathbf{i} = t_a + \mathbf{i}t_b + \mathbf{j}t_c + \mathbf{k}t_d = t \\ t^{\mathbf{j}} &= +\mathbf{j}t\mathbf{j} = t_a + \mathbf{i}t_b + \mathbf{j}t_c + \mathbf{k}t_d = t \\ t^{\mathbf{k}} &= -\mathbf{k}t\mathbf{k} = t_a + \mathbf{i}t_b + \mathbf{j}t_c + \mathbf{k}t_d = t. \end{aligned} \tag{16}$$

From (16), we see that tessarines are auto-involutive, so a widely linear model is possible to the extent that it is defined the same way as for complex numbers [15].

3.3 Computational Cost

Since both quaternions and tessarines are 4-dimensional algebras, tbe computational cost of QLMS and TLMS is the same. In fact, the computation of the filter output requires $4 \cdot 4 \cdot M$ multiplications per sample. The same effort is required in the weight update equation. Overall, the computational cost of QLMS and TLMS is $32 \cdot M \cdot n_{samples}$ multiplications. The situation changes when working with widely linear filters. In tessarine algebra, only the vector $\mathbf{x}_n$ and its conjugate are necessary in the algorithm definition, so whereas the WL-QLMS requires $4 \cdot 32 \cdot M \cdot n_{samples}$ multiplications, the computational cost in the WL-TLMS is reduced to $2 \cdot 32 \cdot M \cdot n_{samples}$.

4 Simulations

In this section, we propose two examples in order to make a performance comparison between quaternion and tessarine filtering according to the input signals. The simulation layout is represented in Fig. 1. In both simulations we have a system $\mathbf{w}_0$ to be identified, which is defined in the time domain by a set of random weights, uniformly distributed in the range $[-1, 1]$.

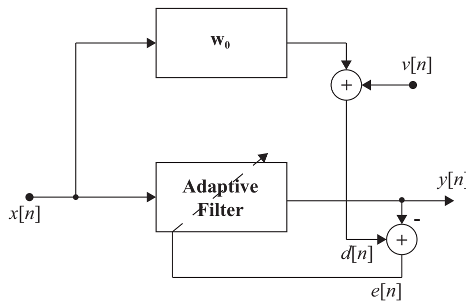

Fig. 1. Simulation layout.

4.1 Generic circular input signals

In this first example, we apply QLMS and TLMS in a context where the input signal $x[n]$ is considered as either a quaternion-valued or tessarine-valued colored noise with unit variance and it was obtained by filtering the white Gaussian noise $\eta[n]$ as $x[n] = bx[n-1] + \frac{\sqrt{1-b^2}}{\sqrt{4}}\eta[n]$, where b is a filtering parameter (here it was chosen as $b = 0.7$). The additive signal $v[n]$ is defined the same way as $x[n]$, but the parameter b is set to zero.

Signal $x[n]$ is circular, all its components are uncorrelated to each other, so, at first glance, it seems to be equivalent to consider it as a quaternion or a tessarine. In effect, our results are concordant with the expectations (Fig. 2): given the same filter parameters ($M = 12$, $\mu = 0.008$), the QLMS and TLMS exhibit the same MSE. In this simulation, after 5000 samples, the weights $\mathbf{w}_0$ change abruptly. The two filters run after the variation with the same rate.

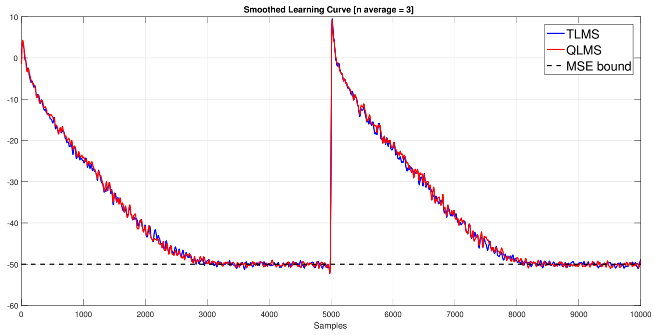

Fig. 2. MSE: QLMS vs TLMS with proper input signal.

4.2 Ambisonic improper audio input signals

The second example we propose in this paper makes use of a well-structured 3D audio input signal. This signal has 4 components which were recorded by 4 microphones in accordance with the 3D audio technique called Ambisonics (B-Format). The first order Ambisonic B-Format technique mounts 4 coincident microphones, orthogonal to one another: one omnidirectional microphone (W) and three figure-of-eight microphones (X, Y, Z). Each microphone signal can be assigned to a 4-dimensional algebra component as

$$x[n] = x_W[n] + x_X[n]\mathbf{i} + x_Y[n]\mathbf{j} + x_Z[n]\mathbf{k}. \tag{17}$$

However, we want to prove that this assignment is not merely a matter of convenience. In fact, Fig. 3 shows how the choice of a different algebra, defining the mathematical space, determines the filter performance. Giving an interpretation of Fig. 3, we understand that a B-Format signal is rather inclined to be represented by quaternion algebra than by tessarines (the QLMS converges faster than TLMS on equal terms). In truth, in previous works [12, 13], the authors of this paper found a relation between the sound field as decomposed by Ambisonics and a quaternion-valued representation. Ambisonics decomposes the sound pressure field into a linear combination of *spherical harmonics* and the subgroup SO(3) of 3D Euclidean rotations does have a representation on the $(2m+1)$-dimensional Hilbert space with spherical harmonics (span $\{Y_{mn}^{\sigma}(\theta,\phi), 0 \leq n \leq m, \sigma = \pm 1\}$, where $Y_{mn}^{\sigma}(\theta,\phi)$ are the spherical harmonics). It is known that the subspace of pure quaternions (those quaternions with null real component) is isomorphic to rotations. That said, the quaternion representation of Ambisonics does not simply consist in a compact formalism, but it has a physical and geometrical meaning.

In our simulation, the source is a monodimensional unit-variance white Gaussian noise in a computer-generated anechoic room. The source was placed at a distance of 20 cm from the B-Format array, 45° off-axis with the X microphone. Additive unit-variance white Gaussian noise $\nu[n] \in \mathbb{H}$, with $n = 0, 2..., P-1$, was summed to the output signal of the system to be identified ($d[n] = \mathbf{w}_0^T \mathbf{x} + \nu[n]$). The filter parameters were chosen as $M = 12$, $\mu = 0.3$.

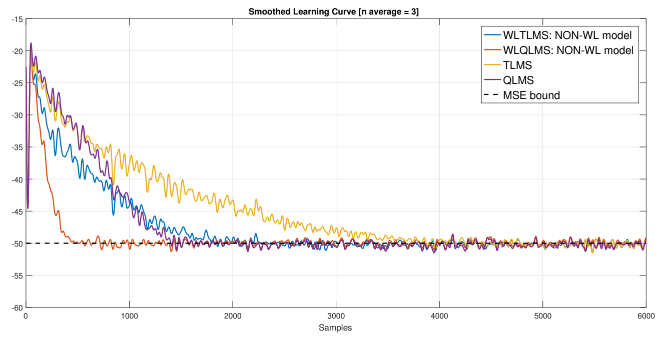

Fig. 3. MSE: (WL-)QLMS vs (WL-)TLMS with Ambisonic improper input signal. *NON-WL model* in the legend box refers to a system to be identified which only has $\mathbf{w}_0$ weights.

In addition, in Section 3.2, we emphasized the possibility to build a widely linear algorithm (WL-QLMS, WL-TLMS). Since the Ambisonic B-Format is improper, we expect the WL-QLMS and WL-TLMS to outperform QLMS and TLMS, respectively. The results from our simulation meet the expectations.

5 Conclusion

Hypercomplex algebras are decisively making their own way in adaptive filtering applications. The question today is whether we really need such hypercomplex models to represent our systems. Besides that, what determines the rejection of one algebra in favor of another? In this paper, we proposed a simple comparison between two 4-dimensional hypercomplex algebras: quaternions and tessarines. We learned from simulations that some systems can be considered either quaternion-valued or tessarine-valued. In other cases, the choice of the algebraic representation determines the performance of the whole system. For instance, we introduced the Ambisonic B-Format signals into a 4-dimensional system and we saw that a quaternion adaptive algorithm converges much faster than its tessarine counterpart. We have found a relation between spherical harmonics/rotations and quaternions. The group of rotations and quaternions are both non-commutative. There is no equivalent in tessarine algebra, which is in fact commutative. However, further investigation may discover environments where tessarine processing is the most appropriate. In a next work, we are going to publish results from simulations in Ambisonic and Uniform Linear Array contexts, where in both cases the signals at the sensors are correlated. We saw that there are geometries in which a faster convergence is reached by means of tessarine algorithms.

References

1. Barthélemy, Q., Larue, A., Mars, J.I.: About qlms derivations. IEEE Trans. Signal Process. Letters 21(2), 240–243 (2014)
2. Gerzon, M.A.: Ambisonics. part two: Studio techniques. Studio Sound 17, 24–26, 28–30 (1975)
3. Gerzon, M.A.: Ambisonics in multichannel broadcastingand video. J. Audio Eng. Soc. 33(11), 859871 (1985)
4. Jahanchahi, C., Took, C., Mandic, D.: A class of quaternion valued affine projection algorithms. Signal Process. (93), 1712 – 1723 (2013)
5. Jahanehahi, C., Took, C.C., Mandic, D.P.: The widely linear quaternion recursive least squares filter. In: Proc. 2nd Int. Workshop Cognitive Information Processing (CIP 2010)
6. Katunin, A.: Three-dimensional octonion wavelet transform. J. Applied Mathematics and Computational Mechanics 13(1), 33 – 38 (2014)
7. Kraft, E.: A quaternion-based unscented kalman filter for orientation tracking. In: Proc. 6th Int. Conf. Information Fusion. pp. 47 – 54 (ISIF 2003)
8. Li, X., Adalı, T.: Complex-valued linear and widely linear filtering using mse and gaussian entropy. IEEE Trans. on Sig. Proc. 60(11), 5672 – 5684 (2012)
9. Mandic, D.P., Goh, V.S.L.: Complex Valued Nonlinear Adaptive Filters: Noncircularity, Widely Linear and Neural Models. Wiley (2009)
10. Neeser, F.D., Massey, J.L.: Proper complex random processes with applications to information theory. IEEE Trans. Inf. Theory 39(4), 1293 – 1302 (1993)
11. Ortolani, F., Comminiello, D., Scarpiniti, M., Uncini, A.: Frequency domain quaternion adaptive filters: Algorithms and convergence performance. Signal Processing 136, 69 – 80 (Jul 2017)
12. Ortolani, F., Comminiello, D., Uncini, A.: The widely linear block quaternion least mean square algorithm for fast computation in 3d audio systems. In: Proc. 26th International Workshop on Machine Learning for Signal Processing (Sep MLSP, 2016)
13. Ortolani, F., Uncini, A.: A new approach to acoustic beamforming from virtual microphones based on Ambisonics for adaptive noise cancelling. In: IEEE 36th Int. Conf. on Electronics and Nanotechnology (ELNANO) (Apr 2016)
14. Picinbono, B.: On circularity. IEEE Trans. on Sig. Proc. 42(11), 3473 – 3482 (1994)
15. Picinbono, B., Chevalier, P.: Widely linear estimation with complex data. IEEE Trans. on Sig. Proc. 43(8), 2030 – 2033 (1995)
16. Rumsey, F.: Spatial Audio. Focal Press (2001)
17. Said, S., Bihan, N.L., Sangwine, S.J.: Fast complexified quaternion fourier transform. IEEE Trans. Signal Process. 56(4), 1522 – 1531 (2008)
18. Shoemake, K.: Animating rotation with quaternion calculus. ACM SIGGRAPH Course Notes
19. Took, C.C., Mandic, D.P.: The quaternion lms algorithm for adaptive filtering of hypercomplex processes. IEEE Trans. Signal Process. 57(4), 1316–1327 (2009)
20. Took, C., Mandic, D.P.: A quaternion widely linear adaptive filter. IEEE Trans. Signal Process. 58(8), 4427 – 4431 (2010)

21. Took, C., Mandic, D.P.: Augmented second-order statistics of quaternion random signals. Signal Process. 91(2), 214 – 224 (Feb 2011)
22. Vakhania, N.N.: Random vectors with values in quaternion Hilbert spaces. Theory of Probability and its Applications 43(1), 18 – 40 (1998)